STABILITY OF FLOW OF A VISCOUS LIQUID DOWN AN INCLINED PLANE

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Using the Navier–Stokes equation the stability of a layer of viscous liquid flowing down a solid surface under gravity is studied in the linear formulation. The effect of surface tension and the inclination of the solid surface on the limits of stability are examined also. Curves are calculated for the neutral stability with respect to two types of perturbations – surface waves and shear waves.

1. Formulation of the Problem

The detailed study of the wave flow of a layer of viscous liquid down an inclined plane began with the papers of Kapitsa and Kapitsa [1, 2]. In [3-6] the problem of the stability of the runoff of a film with a free surface is reduced to the problem of finding the eigenvalues of the Orr–Sommerfeld equation, which enables one to calculate the limits of stability of the parameters on the plane. One of the limits, corresponding to perturbations of the type of surface waves, was found analytically in [3-5] for small αRe. The existence of a second limit corresponding to perturbations of the type of shear waves was first noted in [3], and this neutral curve was calculated in [6] using asymptotically large αRe. The papers [7-10] are devoted to a study of the nonlinear problem. Paper [10] contains an extensive list of references on problems of the nonlinear stability of a falling film.

We consider a layer of viscous incompressible liquid flowing down a plane surface inclined at an angle $0 < \beta \leq 90°$ with the horizontal (Fig. 1). We assume that surface tension σ acts on the free boundary.

We take as units of length, time, and mass d, $d^2\nu^{-1}$, and ρd^3 respectively, where d is the average thickness of the layer, ν is the kinematic viscosity, and ρ is the density. We introduce the dimensionless Reynolds number $\mathrm{Re} = 0.5 \sin \beta g d^3/\nu^2$ and the Weber number $W = \sigma d/\rho\nu^2$. We seek a solution of the hydrodynamics equations in dimensionless form having a period $2\pi/\alpha$ in x. The flow of the liquid is due to gravity. It is clear from the equations of motion that the average of the longitudinal pressure gradient over a period does not depend on y; we assume it is zero:

$$\frac{\alpha}{2\pi}\int_0^{2\pi/\alpha}\frac{\partial p}{\partial x}\,dx = 0.$$

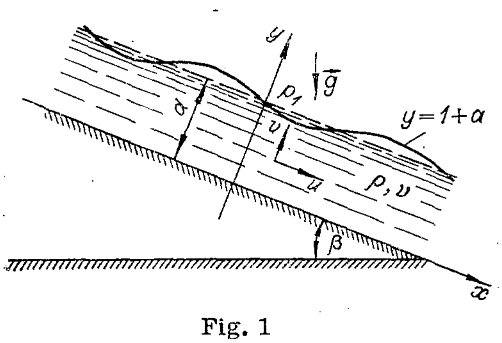

Fig. 1

On the free surface $y = 1 + a(x, t)$ the following dynamic and kinematic conditions are satisfied [11]: $P_{nn} + P_1 = Wa(1 + a^2)^{-3/2}$; $P_{n\tau} = 0$; $a_t + ua_x = v$, where t is the time, $a(x, t)$ is the perturbation of the free boundary, P_{nn} and and $P_{n\tau}$ are the normal and tangential stresses, p_1 = const is the atmospheric pressure; u and v are the components of velocity, vanishing at the solid wall. Under these conditions there is a steady plane parallel flow

$$u_0 = \mathrm{Re}\, U(y);\quad v_0 = 0;\quad p_0 = p_1 + \mathrm{Re}\, U' \operatorname{ctg} \beta \,(U = 2y - y^2) \qquad (1.1)$$

with an unperturbed free boundary $a = 0$.

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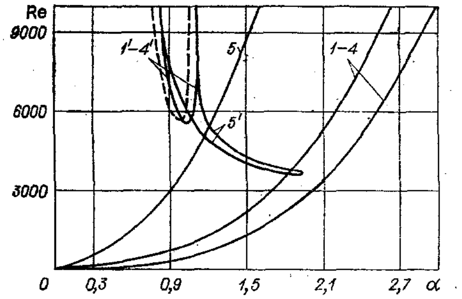

Fig. 2

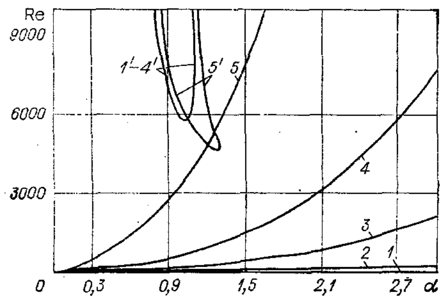

Fig. 3

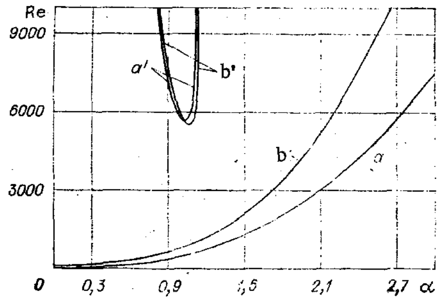

Fig. 4

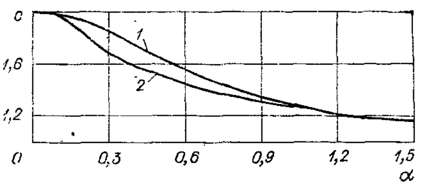

Fig. 5

Linearizing the equations of motion in the Gromek–Lamb form in the neighborhood of the flow (1.1) and finding solutions proportional to exp $i\alpha(x - \mathrm{Re}\,ct)$ we obtain an eigenvalue problem for the parameter c:

$$\begin{aligned} &u' = -\alpha^2 v - \omega;\quad v' = -u;\quad u(0) = v(0) = 0;\quad v(1) = \mathrm{Re}(1-c)a;\\ &\omega' = i\alpha \mathrm{Re}(cu - U'v) - i\alpha h;\quad h' = \mathrm{Re}\,U'u - \alpha^2 \mathrm{Re}\,cv + (i\alpha - \mathrm{Re}\,U)\omega;\\ &h + (2i\alpha - \mathrm{Re})u - (2\mathrm{Re}\,\mathrm{ctg}\,\beta + \alpha^2 W)a = 0;\quad \omega + 2\alpha^2 v + 2\mathrm{Re}\,a = 0,\quad (y = 1). \end{aligned} \tag{1.2}$$

Here u, $i\alpha v$, ω, h, and a are, respectively, the amplitudes of the perturbation of the longitudinal and transverse velocities, the vorticity, the total pressure, and the free boundary.

2. Results of the Calculations

To calculate the neutral curves over a wide range of the parameters (the actual calculations were made for $0 \le \alpha \le 24$, $0 < \mathrm{Re} \le 10^5$) it is convenient to use the differential pivot method proposed in [12].

We introduce the notation

$$q_1 = \begin{pmatrix} u \\ v \end{pmatrix};\quad q_2 = \begin{pmatrix} \omega \\ h \end{pmatrix};\quad A_{11} = \begin{pmatrix} 0 & -\alpha^2 \\ -1 & 0 \end{pmatrix};\quad A_{12} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix};$$

$$A_{21} = \begin{pmatrix} i\alpha\,\mathrm{Re}\,c & -i\alpha\,\mathrm{Re}\,U' \\ \mathrm{Re}\,U' & -\alpha^2\,\mathrm{Re}\,c \end{pmatrix};\quad A_{22} = \begin{pmatrix} 0 & -i\alpha \\ i\alpha - \mathrm{Re}\,U & 0 \end{pmatrix}$$

and write (1.2) in the form

$$q_s' = A_{s1} q_1 + A_{s2} q_2,\quad (s = 1,\ 2). \tag{2.1}$$

Setting $q_1 = G q_2$ in (2.1) and taking account of the nonslip condition we find the 2×2 matrix G(y) as a solution of the Cauchy problem

$$G' = A_{11} G + A_{12} - G A_{21} G - G A_{22},\quad G(0) = 0. \tag{2.2}$$

We give the boundary conditions on the free surface the matrix form

$$B_1 q_1(1) + B_2 q_2(1) + N a = 0;$$

$$B_1 = \begin{pmatrix} 2i\alpha - \mathrm{Re} & 0 \\ 0 & -2\alpha^2 \\ 0 & -1 \end{pmatrix};\quad B_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \\ 0 & 0 \end{pmatrix};\quad N = \begin{pmatrix} -2\mathrm{Re}\,\mathrm{ctg}\,\beta - \alpha^2 W \\ -2\,\mathrm{Re} \\ \mathrm{Re}\,(1 - c) \end{pmatrix}$$

from which, after substituting $q_1(1) = G(1)\, q_2(1)$, we obtain the complex equation

$$F \equiv \det \| M \mid N \| = 0;\quad M = B_1 G(1) + B_2,$$

TABLE 1

No.	Liquid	$\sigma\cdot 10^3$, N/m	ρ, kg/m³	$\nu\cdot 10^6$, m²/sec	$W=\frac{\sigma d}{\rho\nu^2}$
1	—	0	Arbitrary	Arbitrary	0
2	Glycerin	59,4	1 259	117 500	0,0342
3	Ethyl alcohol . . .	22,8	790	220	5 962
4	Water.	72,75	1 000	102	72 750
5	Mercury	460	13 550	11,4	$261\cdot 10^4$

which reduces to a system of two real equations

$$F_r(\alpha,\ \mathrm{Re},\ c,\ \beta,\ W)=0;\ F_i(\alpha,\ \mathrm{Re},\ c,\ \beta,\ W)=0 \tag{2.3}$$

for the parameters on the neutral stability curve (c is real). Equations (2.3) were solved by Newton's method, approximating the partial derivatives by finite differences, and the differential equation (2.2) was integrated numerically by the standard Runge–Kutta method with automatic choice of step. The roots were found by varying α or Re; the implicit functions $c(\alpha)$, $\mathrm{Re}(\alpha)$ or $c(\mathrm{Re})$, $\alpha(\mathrm{Re})$ were calculated for fixed β and W, and the initial approximations for Newton's method were made by Aitken extrapolation. The step in the parameter was chosen automatically depending on the number of iterations necessary to achieve the prescribed accuracy; this saves computing time.

The calculations were made for the values of the Weber number listed in Table 1. The physical constants were taken at 20°C from [13] for an average layer thickness $d=10^{-3}$ m. Figures 2 and 3 illustrate the effect of surface tension on the stability limits. Two kinds of neutral curves are shown; the lower (1, 2, etc.) correspond to perturbations of the type of surface waves, and the upper (1', 2', etc.) to perturbations of the type of shear waves. The curves 1 and 1', 2 and 2' etc. were calculated for a fixed slope (Fig. 2, $\beta=1°$; Fig. 3, $\beta=90°$) and the Weber numbers listed in lines 1-5 of Table 1. For comparison the open curve of Fig. 2 is the neutral curve obtained in [6] by the asymptotic method in the approximation $\alpha\mathrm{Re}\gg 1$ ($W=0$, $\beta=1°$). According to [3] calculations confirmed that for a vertical wall ($\beta=90°$) and $W=0$ the axis $\mathrm{Re}=0$ is the curve of netural stability with respect to surface waves (curve 1 of Fig. 3). Figure 4 shows neutral curves of two types for $w=72750$ (water). The stability limits a and a' are plotted for the angle of inclination $\beta=90°$; δ and δ' are for $\beta=1°$. In accordance with the asymptotically small values of α obtained in [3-5] the neutral curves for surface waves emerge from the point $\alpha=0$, $\mathrm{Re}=1.25\cot\beta$ and, as calculation shows, extend to infinity with increasing α, the more steeply the larger W (or the smaller the angle β). An increase in the surface tension (decrease in the angle of inclination) has the opposite effect on the tongue-shaped netural curves of the second type: the curves drop downward, enabling one to speak of the effect of destabilization. The destabilizing effect is weak for small W but increases with increasing W. For large Reynolds numbers ($\sim 10^5$) the curves of the second family plotted for various values of β and W practically coincide. It is important to note that for large enough W (small enough β) the neutral curves of the two types intersect (Figs. 2, 3) forming a range of wave numbers in which the role of the most dangerous perturbations shifts to shear waves.

The dependence of the phase velocity of surface waves on the wave number is shown graphically in Fig. 5, where the maximum velocity of parallel flow is chosen as a unit. Curves 1 and 2 ($\beta=45°$) correspond to $W=0$ and $2.35\cdot 10^6$. Calculations show that the velocity of propagation of shortwave perturbations is slightly different from unity – the value of the velocity of the primary flow at the free surface (e.g. on curve 1 for $\alpha=23.5$, $c=1.0007$). A similar result was found in [14] for another problem with a free boundary.

In experiments with water and ethyl alcohol [2], in particular, the critical Reynolds number was found below which wave behavior for the runoff of a liquid film does not develop. Measurements for water ($W=7400$) gave $\alpha=0.092$, $\mathrm{Re}=8.06$, and for alcohol ($W=1145$) $\alpha=0.143$, $\mathrm{Re}=5.02$. The critical Reynolds numbers calculated on BÉSM-4 and M-222 computers for wave numbers taken from the experimental data were 6.35 for water and 3.97 for alcohol, which are somewhat below the experimental critical values.

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